Bringing non-individuality and non-contextuality together

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Introduction

Given any quantum state of a system with observables A, B, C, is there a way to respectively assign numerical values $v(A)$, $v(B)$, $v(C)$ to those observables?

Plan of the talk

- The Kochen-Specker (KS) theorem
- Non-individuality and a way out of the KS contradiction (de Barros, Holik and Krause (BHK))
- Objections to BHK
- Final remarks

Hilbert space structure

To any self-adjoint operator $\hat{A} : \mathcal{H} \to \mathcal{H}$ in an n-dimensional Hilbert space $\mathcal{H}$ there is an orthonormal basis $\{|a_i\rangle\} \subseteq \mathcal{H}$ such that

$$\hat{A} = \sum_i a_i \hat{P}_i,$$

where $\hat{P}_i = |a_i\rangle\langle a_i|$ are projection operators (projectors) with eigenvalues 0 or 1.

$\Rightarrow$ $\hat{P}_i$ are associated to propositions!

Valuation functions

A global valuation function v satisfies

$(*)$ *value-rule*: $v(\hat{A}) \in \mathbb{R}$ belongs to the spectrum of $\hat{A}$;

$(\dagger)$ *functional composition principle*: $v(h(\hat{A})) = h(v(\hat{A}))$ for any real-valued function h.

Therefore,

$$v(\hat{A} + \hat{B}) = v(\hat{A}) + v(\hat{B})$$

for commuting operators $\hat{A}$ and $\hat{B}$.

Value assignment to projectors

A set $\{\hat{P}_i\}$ of pair-wise commuting projectors $[\hat{P}_i, \hat{P}_j] = 0$ forms a resolution of the identity

$$\sum_i \hat{P}_i = \hat{\mathbb{1}}.$$

It implies at once that

$$v(\hat{\mathbb{1}}) = v\left(\sum_i \hat{P}_i\right) = \sum_i v(\hat{P}_i) = 1.$$

The KS theorem

Given operators $\hat{A}$, $\hat{B}$, $\hat{C}$, is there a consistent way to respectively assign numerical values $v(\hat{A})$, $v(\hat{B})$, $v(\hat{C})$ to those operators?

$\Rightarrow$ No.

Theorem (Kochen-Specker): *no such valuation function* v *exists whenever* $\dim(\mathcal{H}) > 2$.

Proof: Cabello's 18-vectors proof (next frame).

18-vectors proof (1/2)

Suppose $\dim(\mathcal{H}) = 4$ and projectors $\hat{P}_{ijk\ell}$ $(i, j, k, \ell = 0, \pm 1)$.

The vector $|\psi\rangle = (0, 0, 0, 1)$ associated to $\hat{P}_{0001}$ is orthogonal to $|\varphi\rangle = (0, 0, 1, 0)$, the latter associated to $\hat{P}_{0010}$. Hence,

$$[\hat{P}_{0001}, \hat{P}_{0010}] = 0.$$

The same with $\hat{P}_{1100}$ and $\hat{P}_{1\text{-}100}$ and so on. It follows that, for instance,

$$\hat{P}_{0001} + \hat{P}_{0010} + \hat{P}_{1100} + \hat{P}_{1\text{-}100} = \hat{\mathbb{1}}.$$

Notation: $v(\hat{P}_{ijk\ell}) \triangleq v(ijkl)$.

18-vectors proof (2/2)

$$\begin{aligned}
v(0001)+v(0010)+v(1100)+v(1\text{-}100) &= 1 && (1)\\
v(0001)+v(0100)+v(1010)+v(10\text{-}10) &= 1 && (2)\\
v(1\text{-}11\text{-}1)+v(1\text{-}1\text{-}11)+v(1100)+v(0011) &= 1 && (3)\\
v(1\text{-}11\text{-}1)+v(1111)+v(10\text{-}10)+v(010\text{-}1) &= 1 && (4)\\
v(0010)+v(0100)+v(1001)+v(100\text{-}1) &= 1 && (5)\\
v(1\text{-}1\text{-}11)+v(1111)+v(100\text{-}1)+v(01\text{-}10) &= 1 && (6)\\
v(11\text{-}11)+v(111\text{-}1)+v(1\text{-}100)+v(0011) &= 1 && (7)\\
v(11\text{-}11)+v(\text{-}1111)+v(1010)+(010\text{-}1) &= 1 && (8)\\
\underbrace{v(111\text{-}1)+v(\text{-}1111)+v(1001)+v(01\text{-}10)}_{\text{even}} &= \underbrace{1}_{\text{odd}} && (9)
\end{aligned}$$

Ways out of the KS contradiction

(i) properties do not have well-defined values (the traditional way)

(ii) properties or particles may be indistinguishable (BHK)

Let's take (ii).

Quasi-set theory $\mathcal{Q}$

$\mathcal{Q}$ is a ZFU-like set theory with two kinds of atoms:
m-atoms m (quantum) and M-atoms M (classical).

Collections of atoms, called q-sets Q, are not atoms.

m-atoms may be indistinguishable ($\equiv$) but have no identity ($=_E$):

Definition 1. $x =_E y \leftrightarrow (Q(x) \wedge Q(y) \wedge \forall z(z \in x \leftrightarrow z \in y)) \vee (M(x) \wedge M(y) \wedge \forall z(x \in z \leftrightarrow y \in z))$

"Identical objects are either q-sets containing all the same elements or M-atoms pertaining to the same q-sets."

Weak pairs and unitary q-sets

Weak pairs: given any q-set z, we may form the q-set $[x, y]_z$ of the indiscernibles of either x or y that belongs to z.

If $x \equiv y$, then $[x, y]_z$ reduces to the q-set $[x]_z$ (indiscernibles from x that belongs to z).

Cardinality: the quasi-cardinality qc attributes to each q-set its quantity of elements.

One can prove in $\mathcal{Q}$ that $[x]_z$ has a subqset with quasi-cardinal $qc = 1$, denoted $[[x]]_z$ (the *strong singleton* of x).

The BHK solution to the KS contradiction (1/2)

(ii) properties or particles may be indistinguishable (BHK)

Indistinguishable particles and indistinguishable properties are being measured in distinct contexts:

> [the paradox] *can be avoided, for the contradiction assumes that "the same" properties are measured in "the same" particles in different contexts. However, if we realize that we measure indistinguishable properties over indistinguishable particles there will be no surprise in acknowledging that the obtained results may differ.*
> (BHK 2017 p. 3)

The BHK solution to the KS contradiction (2/2)

> *This is the fundamental point: we have indistinguishable properties [...] and indistinguishable particles. Take a collection (qset) os such properties: this is a context. We may form several contexts this way. Take a particle and one context and measure the corresponding properties: we have outcomes. Now take another "indistinguishable" context and an indistinguishable particle. Although the properties and the particles are indistinguishable, the outcomes may be different.* (BHK 2017, p. 8)

$\Rightarrow$ Indistinguishable non-individuals are present in distinct acts of measurement.

The BHK solution: propositions in $\mathcal{Q}$

Consider again the projector $\hat{P}_{0010}$ associated to property P_{0010}. A proposition in $\mathcal{Q}$ attributing such property to a particle x is written as

$$\big\langle \langle P_{0010}; 1 \rangle; [[x]]_z \big\rangle.$$

Taking $[[x']]_z$ as an indiscernible from $[[x]]_z$, it makes perfect sense to say that

$$\big\langle \langle P_{0010}; 0 \rangle; [[x']]_z \big\rangle.$$

The BHK solution: propositions in $\mathcal{Q}$ (more generally...)

Consider $M_A(B)$ to mean "the value of A lies in B" and q a system of interest. Classically, q has identity and

$$\big\langle\langle M_A(B); 1\rangle; q\big\rangle \wedge \big\langle\langle M_A(B); 0\rangle; q\big\rangle$$

gives rise to the KS contradiction.

However, if q is a non-individual represented by its strong singleton, no KS contradiction follows from

$$\big\langle\langle M_A(B); 1\rangle; [[q]]_z\big\rangle \wedge \big\langle\langle M_A(B); 0\rangle; [[q]]_z\big\rangle.$$

Difficulties to BHK

- Methodology
- Consistency
- Formalism

Methodological difficulties

Indistiguishability (permutation symmetry) plays no role in the BHK solution, since both the former and the KS contradiction are features of quantum mechanics itself.

Lack of identity (non-individuals) is the one doing the job:
a metaphysical thesis is playing a major role in avoiding a physical theorem. Difficulties:

- not equivalent to quantum mechanics with individuals;
- not naturalistically acceptable;
- quantum mechanics actually plays no role in the BHK solution.

Consistency

What is non-individuality really avoiding?

> *[I]f we prepare N copies of a spin $1/2$ system and measure the spin in the same direction, say S_z, we can obtain a distinguishable series of results. As an example with $N = 5$, we may obtain $(1/2; 1/2; -1/2; -1/2; 1/2)$. However, here it comes the interesting part: while all preparations and measurements are essentially equivalent (i.e., indistinguishable), they are not the same ones in the sense of being just one.* (BHK 2017, p. 15)

$\Rightarrow$ KS is not about measurement of the same quantity time after time, in the same setting. Rather, it is about the distribution of values to incompatible quantities simultaneously.

Consistency: property attribution

If the KS contradiction is really avoided, then there exists a global valuation function attributing values to observables irrespective of contexts.

It follows that

$$\big\langle \langle M_A(B); 1\rangle; q \big\rangle \wedge \big\langle \langle M_A(B); 0\rangle; q \big\rangle,$$

i.e., any particle could have the observable with that value, and also no particle has the observable with that value.

$\Rightarrow$ Property attribution ends up meaningless.
$\Rightarrow$ The idea of incompatible observables seems to be lost.

Consistency: contexts

Suppose it makes sense to speak of incompatible contexts: how do we define them?

The fact that in distinct contexts the same observable may have incompatible values is not obtained by non-individuality, but is the very claim that observables are contextual, a consequence of KS.

What BHK really accounted for was the temporal repetition of similar acts of measurement, not a liberation of a globally defined valuation function.

Formalism

BHK seems to be incompatible with $\mathcal{Q}$.

Definition (Quasi-function). *f is a quasi-function among q-sets A and B if and only if f is a quasi-relation between A and B such that for every $u \in A$ there is a $v \in B$ such that if $\langle u, v \rangle \in f$ and $\langle w, z \rangle \in f$ and $u \equiv w$ then $v \equiv z$.*

That is: no quasi-function may attribute distinct values to indiscernible elements, in particular, we cannot have $v(M_A(B), [[q]]_z) = 1$ and also $v(M_A(B), [[q']]_z) = 0$.

$\Rightarrow$ Indiscernible entries should be mapped into indiscernible values.

Final remarks

- Due to the difficulties we have pointed to, it seems the BHK solution fails in avoiding the KS contradiction.
- The metaphysics of non-individuality does not lead to violations of KS
- Quasi-set theory cannot represent an account of properties that goes against quantum mechanics, and that is good!

thank you

obrigado

gracías

References

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