The Fate of Bundle and Substratum Theories Under KS Theorem

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Abstract

Basically, *bundle theories* and *substratum theories* are metaphysical accounts of the following features of concrete particulars: i) their constitution from more basic entities, ii) their individuality, and iii) their possession of properties. When connected to quantum mechanics, most discussions focus on the problem of individuality of quantum particles. In this chapter, we shall argue that irrespective of how those theories fare on *this* task, both fail on accounting for the constitution and property possession when quantum mechanics enters the stage. The Kochen-Specker theorem of quantum theory strikes directly against the account of property possession and constitution provided by the current versions of those metaphysical theories. Two major claims shall result from our investigations. First, that a revision on the nature of particular entities will have to be advanced in case one still wishes to hold that quantum entities are particulars. Second, that claims concerning metaphysical underdetermination between individuality and non-individuality will have to be revised in the light of the restrictions imposed by the Kochen-Specker theorem.

Keywords: constitution; bundle theory; substratum; Kochen-Specker theorem.

1 Introduction

Two of the most familiar approaches to the nature of particular concrete objects in metaphysics are the bundle theory and the substratum theory (see Loux 2006; French and Krause 2006, chap. 1).[1] Both theories are thought to account for three related features of such particulars:[2]

i) their constitution in terms of more basic entities,

ii) their individuality, explained in terms of their constitution and in terms of their ingredients, and

[1]These are not the only theories available, of course, but we shall confine ourselves to these theories in this paper.

[2]For the sake of brevity, whenever we mention a particular in this paper, it will be understood that it is a concrete particular, unless stated otherwise.

iii) the predicational nexus, accounting for how is it that a particular may be said to bear or have a property.

Roughly, *bundle theories* conceive of concrete objects as being entirely constituted by the properties they instantiate, from which it results that objects are nothing but a bundle or cluster of their co-instantiated properties. On the other hand, *substratum theories* conceive of concrete objects as being constituted by the particular's instantiated properties and by a further ingredient, a particular of non-qualitative nature, a *self-individuating substratum* or *bare particular*. Sometimes, instead of positing a particular object such as a substratum, it is also proposed that each object has only properties involved in its constitution, but that a special property of a non-qualitative nature, a *haecceity* or *individual essence*, is also present. On this account, each object has its own haecceity, which accounts for its individuality, and this makes it more similar to the substratum approach than to the bundle approach (and that is why we shall treat it so too). Both theories account for the fact that an object has a property by claiming that the property constitutes the object in some sense; their main difference concerns the individuality of particulars: is the individuality of a particular accounted solely by their qualitative features, or is a further non-qualitative ingredient needed?

Just to be sure: the problem of individuality is a metaphysical problem. It concerns explaining what is it that makes a particular object that object that it is, distinct from everything else (for a careful discussion of the terminology associated with individuality, see Krause and Arenhart 2018). Bundle theories advance the claim that properties are enough for that. Substratum theories suggest the need for a special ingredient. This is specially important, because so far, the dialectics of the debate between these theories concerns mainly the issue over the problem of individuality. For instance, substratum theorists will quickly point out an intimate relation between bundle theory and the highly controversial *Principle of the Identity of Indiscernibles* (PII), whereas bundle theorists will claim an incoherence on the very idea of an elusive substratum, lying beyond all properties, constituting objects. Either way, no consensus has been achieved and the nature of concrete particulars continues to raise important issues within metaphysics. We believe, however, that fundamental physics is welcome in the debate and may bring some ideas to enrich and, potentially, enlighten the controversy. Our claim is that physics has important lessons for those theories on what concerns property possession, not only individuality.

This shift of focus is important, because in Quantum Mechanics (QM), the *state* of a physical system is labeled by a vector pertaining to a vector space. Traditionally, the same vector is to be regarded as a description of the physical object under study. Metaphysically speaking, this object could, *in principle*, be viewed through the lenses of both bundle and substratum theories (given that those theories should apply to every object). Nonetheless, it has been argued that, because of the commitment of bundle theories with the truth of the PII, a false instance of the latter would result in the falsehood of the former. As a matter of fact, QM is said to deal with situations in which the stronger versions of PII are demonstrably false, hence implying that bundle theories should be rejected in favor of a substratum framework in the context of quantum individuality (see French and Krause 2006, chap. 4). Yet, this is only part of the story. We will argue, following recent work done on invariance of quantum states and the Kochen-Specker (KS) theorem (de Ronde and Massri 2016), that *both* bundle and substratum theories fail to provide a satisfactory account of quantum mechanical entities. These results will drive us into further discussions regarding the nature of quantum objects and steer us towards a revision over the customary ways of understanding the metaphysics of concrete particulars. Also, this will allow us to sidestep typical discussions focusing on the role of the Identity of Indiscernibles in these discussions and extract some lessons concerning metaphysical underdetermination in quantum mechanics.

The paper is structured as follows: in section 2, we delve into the two metaphysical theories of objects under study here and how they are related to Classical Physics. In section 3, we introduce key aspects of QM, and the KS theorem enters the scene to set the stage for our arguments. Section 4 brings together the ideas fomented in the previous sections and establishes a quantum mechanical attack on both bundle and substratum theories. We finish our paper in section 5, discussing further problems, specifically aiming at the current understanding of underdetermination of metaphysics by physics in light of our findings.

2 Metaphysical theories of particulars and Classical Physics

2.1 Bundle and substratum theories

As mentioned in the introduction, we shall focus on two perspectives related to the understanding of concrete particulars. Both theories are concerned with constituting particulars from more basic ingredients. In this sense, both approaches deal with the problem of providing for the nature of a particular in terms of more basic entities that constitute the particulars; in this sense, these are reductive approaches: concrete particulars are somehow reduced to more basic ingredients that are metaphysically more fundamental, so that concrete particulars need not be part of the fundamental inventory of reality. Demirli (2010, p. 2) explains as follows the problem of constitution:

> In answering the internal constitution question, we may begin an inquiry about the various categories that go into the composition of individual substances and hope that at the end of this inquiry we will come up with a list of ingredients that constitute various individuals. Just as a certain recipe in a cook book provides us with a list of ingredients and instructions for mixing these ingredients together, we may maintain that the list or the recipe of individual substances — God's recipe book, so to say — will tell us what items from various categories are used, and how these items are combined.

Of course, unless one specifies what 'constitution' and 'composition' mean, the problem and any of its answers will remain on a very abstract and perhaps metaphorical level. Typically, 'constitution' is specified in other terms, such as set theoretic (a concrete particular is a set whose members are its ingredients) or mereologic (a concrete particular is a mereological sum of its ingredients), each with its own problems, but we shall not be concerned with specifying any particular version of constitution here (see discussion and further references in Benovsky 2016 and in Jago 2018). What is relevant for us is that the approaches with which we shall be concerned here are primarily approaches to the question of what goes on in the constitution of a particular.

Bundle and substratum theories alike conceive of objects as constituted or composed by properties instantiated by the particulars. In this sense, it is common to understand properties either as universals or as tropes, and each option will give rise to distinct versions of both bundle theories and substratum theories. We shall continue to speak in neutral terms of properties, leaving it open whether these properties should be conceptualized as universals or as tropes. Each version has its own problems and virtues, but it will not be relevant for our purposes whether one or another route is taken (for a recent account of the discussion, see Benovsky 2016). Besides agreeing that properties constitute particulars, bundle theories and substratum theories differ, however, because while bundle theorists claim that the properties exhaust the constitution of the particular, the substratum theorists posit a further underlying ingredient constituting those particulars.

That difference may be put as follows: the bundle theorist wishes a *one category metaphysics, only properties are fundamental*, and they are employed to constitute particulars. Following Loux (2006, p. 107), this is encapsulated by the principle BT:

BT Necessarily, for any concrete entity a, if for any entity, b, b is a constituent of a, then b is an attribute.

The substratum theorist is not willing to embrace this one category metaphysics, and adds a further ingredient (rejecting BT), a substratum, which is a self-individuating particular, not described in terms of qualities. Perhaps substratum theory (ST) could be defined as follows:

ST Necessarily, for any concrete entity a, if for any entity b, b is a constituent of a, then b is either an attribute or else b is a bare particular, and, if b is a bare particular, then b is unique in the constitution of a.

The main reason for adding such a further ingredient in ST concerns individuality and property bearing. It is typically thought that BT is committed to a version of the Principle of the Identity of Indiscernibles (PII):

PII Necessarily, for concrete objects a and b, if for any property P, $(P(a) \leftrightarrow$

$P(b)$), then $a = b$.[3]

However, as argued by Rodriguez-Pereyra (2004), the bundle theory BT may live without PII, or it may even be compatible with the falsity of PII, provided that the bundles are understood as bundles of universals. In this case, one may distinguish a bundle (as a pack of universals) from its numerically distinct instances (see also Rodriguez-Pereyra 2004 for further options on defending bundle theory, as well as Demirli 2010; a further version of the bundle theory is defended in Jago 2018). We need not enter into these controversies, given that we shall not be primarily concerned with the problem of individuality of particulars. As we have mentioned, the dialectics of the debate between bundle theory and substratum theory focuses mainly on the issue of individuality. The possibility of a bundle accounting for scenarios such as Max Black's (1952) two sphere world,[4] or of quantum mechanics' indiscernible particles, are called forth, and the substratum theorist claims that a further ingredient is always needed (for the case of quantum entities, see French and Krause 2006, chap. 4; Arenhart 2017).

As we mentioned, we shall focus mostly on the explanation that each of these theories provide for property exemplification. Recall that both theories agree on how to account for this: a property is possessed by a concrete particular provided that the property is part of the composition of the particular (*i.e.*, is an ingredient of the particular, using Demirli's metaphor). As Jago (2018, p. 3) puts it for the case of bundle theory, the Property Possession for Bundles (PPB) is explained thus:

PPB A concrete particular a possesses property P if, and only if, P is a member of the a-bundle.

Given that 'being a member of the a-bundle' is an explanation of the very idea of constitution, this may be generalized to take into account also the case of the substratum approach. We shall call it simply the 'Property Possession' (PP):

PP A concrete particular a possesses property P if, and only if, property P constitutes a.

From PP the more specific PPB follows when one specifies that a is a bundle of properties and properties constitute such entities by being members of the bundle. Also, from PP it follows (assuming basic logical inferences) that *whenever a property does not constitute a concrete particular, it is not possessed by that*

[3]One could work with at least three different versions of the PII, each of it takes the set of relevant properties to mean something slightly different. For instance, the weakest form, PII(1), states that it is impossible for two individuals to possess all properties and *relations* in common; the next strongest, PII(2), precludes spatio-temporal properties from its description; and the strongest of all, PII(3), encloses only monadic, non-relational properties. Further distinctions could be provided for stronger versions of the PII, such as restrictions to intrinsic properties, or perhaps for what are called 'pure' properties. For a discussion, see Adams (1979).

[4]In this example, recall, two isolated iron spheres, separated two miles apart from each other and indistinguishable in all their properties are, in fact, seen to be numerically distinct.

particular. In this sense, one could claim that insofar as our two target theories adhere to a form of PP, it will be metaphysically determined, for each property P, whether P is possessed or not for each particular a. That means that it is determined, for each particular, by the very nature of the particular, whether a property is possessed or not possessed by that particular. Mittlestaedt (2009, 2011) called that the 'Principle of Thoroughgoing Determination' (PTD):

PTD Given a property P and a concrete particular a, either a possesses P or else it does not posses P (that is, a possesses the complement of P, which we shall denote by $\neg P$).

In other words, if a list of every property P available for an object a could be provided, one could, at least in principle, determine for each of such properties whether a has P or does not have P.

As we have mentioned, we shall shift the focus from individuation to the bearing of properties. We shall operate on a very general level (not assuming anything about the nature of the properties — whether they are tropes or universals — and not assuming any specific account of constitution — set theoretic, mereologic, or other). Our claim will be that, insofar as these theories adhere to PP and, consequently, to PTD, they are ruled out by QM. We begin, however, by making a detour in Classical Physics, in order to illustrate how well these principles work there.

2.2 The way to Classical Physics

One could argue that the bundle approach was favored by Classical Physics because some aspects of the latter made clearer the role of properties constituting objects: its formalism centers around the main idea of physical quantities. Although defensible, such position would undermine the metaphysical details just discussed. It is uncontroversial that Classical Physics takes some form of objective (*i.e.*, preexistent) approach to physical observables, indeed making it a cornerstone to the metaphysical underpinnings of physical theories prior to the 20th century. In the classical framework, properties gain the status of definite physical quantities, that is, well-defined values on a certain interval of the real line $\mathbb{R}$ (*e.g.*, the mass m of a billiard ball, its charge q, the components x^j of its position vector $\mathbf{x}$, so forth).[5] Consequently, measurements are unproblematic and provide us with nothing but a *revelation* of the physical system's objective, actual properties. An object could then be exhausted into an amalgamation of definite, real values of physical observables through which Classical Physics asserted us the object did possess beforehand.

One encodes physical observables into the mathematical notion of real-valued functions over a space $\mathcal{S}$ of *states* of a physical system, with the understanding that, at any given time, a unique member $s \in \mathcal{S}$ can be assigned to the system. Hence, to each physical observable A there corresponds a function $f_A : \mathcal{S} \to \mathbb{R}$

[5] Note that mass and charge are both examples of *internal* or *intrinsic* physical quantities: they refer to the constitution of the thing itself; whereas position, velocity, etc, are *external* physical quantities whose values depends on a given spacetime framework.

such that $f_A(s)$ is the value which A possesses when the state of the system is s.[6] Also, the state s' of a system at time t' is determined *uniquely* by the state s at any earlier time $t < t'$ via dynamical maps

$$\begin{aligned} T : \mathcal{S} &\to \mathcal{S} \\ s &\mapsto s' \,; \quad \forall\, s, s' \in \mathcal{S}, \end{aligned} \tag{1}$$

essentially bringing determinism and the principle of causality to the classical picture. If $\mathcal{S}$ is the state space of an N-particle system, then the state is labeled by points $(q_1, q_2, \ldots, q_{3N}; p_1, p_2, \ldots, p_{3N}) \equiv (q_k, p_k)$ in a $6N$-dimensional phase space, where q_k and p_k are the respective canonical coordinates. The dynamical law is then given by the Hamiltonian $H(q_k, p_k)$ and the canonical equations

$$\frac{\mathrm{d}q_k}{\mathrm{d}t} = \frac{\partial H}{\partial p_k}\,, \quad \frac{\mathrm{d}p_k}{\mathrm{d}t} = -\frac{\partial H}{\partial q_k}\,; \quad k = 1, \ldots, 3N. \tag{2}$$

The equations in (2) are first-order differential equations and hence determine uniquely the state (q_k, p_k) at any time t (provided we have the system's initial conditions), therefore satisfying the requirements of the dynamical maps in (1).

The objectivity feature of properties, that is, the observer-independent character of Classical Physics is captured nicely by the notion of *invariants*: physical quantities having the same value for any reference frame. The transformations between different frames of reference have the mathematical property of constituting a *group*. Since the laws of both classical mechanics and special relativity theory are invariant against the transformations of the Galilei and Poincaré group, respectively, it follows that, in Classical Physics, both *nomological* (static) and *dynamical* properties are consistently translated between distinct frames of reference and, additionally, it allows one to speak meaningfully of independent, preexistent physical observables: an actual state of affairs (ASA).[7]

Let f_A be the value of an observable A in the state $s \in \mathcal{S}$ and $f_A \in \Delta$, where $\Delta \subset \mathbb{R}$ is a Borel subset that represents the real-valued interval of an observable's values. Then, equipped with the usual logical operators $\wedge$, $\vee$, $\neg$ and $\preceq$ for material implication, it is instructive to construct the mapping

$$\Phi : \mathcal{B}(\mathbb{R}) \to L_C \tag{3}$$

from the Borel subset $\mathcal{B}(\mathbb{R})$ onto the Boolean lattice L_C of propositions $P_i \in L_C$ such that $P_i = 0$ or $P_i = 1$, effectively creating a whole structure of definite yes-no (true- or false-valued) propositions corresponding to the presence or absence, respectively, of physical observables' values (properties) associated with a system in a state s. In other words, *a version of the PTD is justified*!

[6] Strictly speaking, f_A is a Borel function and, depending on the observable A, it may be required to be measurable, or continuous, or smooth. Further, the set $\mathcal{S}$ (also known in classical statistical physics as the *space of microstates*) is a symplectic manifold which, in particular, corresponds to the usual phase space of a one-dimensional point particle when $\mathcal{S} = \mathbb{R}^2$ (Döring and Isham 2010).

[7] We are adopting the terminology advanced by de Ronde and Massri (2016).

Here is where classical logical structure (*i.e.*, Boolean logic) gets incorporated into the mathematical structure of Classical Physics, and one can actually talk of a given proposition P_i as an assignable property. Further, one is now able to coherently define an *individual object* by means of the dynamical maps $T(s)$ and the invariance of definite properties P_i (nomological and dynamical): any state $s \in \mathcal{S}$ labeling a collection of properties P_i is uniquely identified, and continually re-identified through its dynamical trajectory in the classical state space $\mathcal{S}$. Moreover, the notion of an individual persists even when one is dealing with indistinguishable physical systems, only this time the argument relies on either a principle of impenetrability (non-overlapping of spatio-temporal trajectories) or particle permutation via Maxwell-Boltzmann statistics.[8] Based on that, it becomes clear that one could make sense of the individuality of classical particles within the background of either a weak version of the PII (allowing spatio-temporal properties to account for the qualitative distinction) or through something that goes beyond all properties, a substratum. So, although it is true that Classical Physics concentrates around the idea that physical observables are objectively determined for classical systems, it is neither obvious nor compelling to assert that its formalism is committed to bundle theory. Admittedly, Classical Physics seems merely to somehow suggest the individuality of physical systems, independently of whether that individuality is achieved through bundle or substratum theories of objects. Indeed, the present situation constitutes a dilemma known as the *underdetermination of metaphysics by physics*, in which a physical theory formalism is compatible with two or more *metaphysical theories*; in our case, a classical physical system could be individuated by both bundle and substratum theories of objects, but no *physical argument* can be made in favor or against one or the other.

2.3 Bundle and substratum theories (revisited)

On what concerns some attempt to look for help from physics on deciding which is best, bundle or substratum, it appears we are back to square one: Classical Physics does not have the resources to decide between bundle or substratum frameworks. It does satisfy the PTD, which both theories do endorse, and even more: on what regards the individuality of classical indistinguishable systems, one is left between the weakened form of PII allowing for spatial properties to account for individuality or the substratum's elusiveness. The choice for which is better or more appropriate requires a digression into controversial metaphysical issues, and not more investigation into physics. The latter is silent about the relevant issues, and that is why the situation we face is called 'the underdetermination of metaphysics by the physics'. One could try to break the underdetermination by claiming that bundle theory, allowing for spatial properties to

[8] Physical systems, such as particles, are said to be indistinguishable in that they possess the same state independent (intrinsic) properties (mass, charge and spin — the nomological properties in de Ronde and Massri 2016). Given two particles A and B, and two *microstates* 1 and 2, particle permutation on the states yields different outcomes regardless of A and B being indistinguishable, hence, A and B are individuals (French and Rickles 2003).

account for individuality, is a far more economic proposal, avoiding substratum (this is suggested in French and Krause 2006, chap. 2). However, that kind of claim does not come form a necessity of physics itself, and as such, will not move the substratum theorist, which has her own battery of *metaphysical* arguments against the bundle theory.

It should be clear by now that both bundle and substratum theories are *distinct* metaphysical positions regarding the nature of concrete particulars; but it would be a mistake to fully embrace their divergences and, in the process, neglect the common ground they stand. Here is why: prior to their farewell, bundle and substratum theorists would concede in mutual agreement on which properties a given concrete particular instantiate; they would always converge on which properties to assign to a certain object (*i.e.*, which corresponding propositions P_i are well-defined and attached to a system). The bifurcation only happens when one of them stays true to the cluster of properties assigned to an object, whereas the other decides to go beyond the mere qualities observed and proposes a new particular ingredient gluing them all together, establishing, at last, an *individual* through the very existence of that ingredient. It is at this moment that one can say with certainty that the bundle and substratum theorists are each speaking their own language. Yet, no matter how foreigner one's dialect may sound to the other, both share the same proto-language: each of them is committed to the basic operation of, initially, assigning a collection of definite properties to a given system; a set of well-defined propositions P_i (this is the PTD at work, of course).

In fact, such fairly overlooked move contains the key to our main argument against bundle and substratum theories; an argument that goes beyond the usual controversies surrounding the PII or the substratum's transcendence by looking at the conceptual frameworks both theories are founded on, namely, the PTD. However, to materialize such endeavor, we shall bring quantum theory and, in particular, the KS theorem into the discussion. Therefore, we now turn our attention to QM.

3 Quantum theory and the KS theorem

3.1 The way to QM

Quantum theory was born out of a body of empirical results and assumptions that were gathered in the early 20th century and systematically organized by the end of the century's first quarter. The founding fathers of QM laid out a mathematical structure that, more than a hundred years later, is still able to match theory and experiment with the utmost precision. But, despite experimental success, on what regards our understanding of the theory, a hundred years have elapsed with no scientific nor philosophical consensus on what exactly quantum theory is talking about; that is, its ontology and its metaphysical counterpart are still an open question.

The general structure of QM may be approached through the lenses of a 'minimal interpretation', a pragmatic approach that takes quantum theory as an algorithm for predicting the probabilistic distributions of *measurement* outcomes done on suitably prepared copies of a given physical system. Although not stressed as fundamental, the probabilities are interpreted as the *relative frequencies* of possible outcomes if corresponding measurements were to be repeated a sufficiently large number of times. In sharp contrast to Classical Physics, nothing is said about whether a system possesses values for a physical observable prior to its measurement.

Traditionally, the minimal interpretation is mathematically translated via complex Hilbert spaces that contain all possible quantum states of a system. It may be a finite- or an infinite-dimensional space, but for our purposes we will be mainly dealing with the former structure. To a given physical system we associate a *separable* Hilbert space $\mathcal{H}$, such that *normalized* vectors $|\psi\rangle \in \mathcal{H}$ correspond to the states of the system.[9] To compose multiple quantum mechanical systems, say N particles, we employ the tensor product between all the respective Hilbert spaces of each system, that is, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$. Accordingly, the corresponding composite quantum state $|\psi\rangle \in \mathcal{H}$ is the tensor product of all $|\psi_i\rangle \in \mathcal{H}_i$,

$$|\psi\rangle = \bigotimes_{i=1}^{N} |\psi_i\rangle. \tag{4}$$

Physical observables are represented by *self-adjoint* linear operators on $\mathcal{H}$. The expected result of measuring an observable A in a state $|\psi\rangle \in \mathcal{H}$ is given by $\langle\psi|\hat{A}|\psi\rangle$, where $\hat{A}$ is the corresponding operator of A. The *spectral theorem* guarantees that, to any self-adjoint operator $\hat{A}$ in an n-dimensional Hilbert space $\mathcal{H}$, there exists an *orthonormal* basis $\{|a_1\rangle, |a_2\rangle, \ldots, |a_n\rangle\} \subseteq \mathcal{H}$ consisting of *eigenvectors* of $\hat{A}$ such that

$$\hat{A} = \sum_{i=1}^{n} a_i |a_i\rangle\langle a_i| \,; \quad a_i \in \mathbb{R}. \tag{5}$$

The values a_i constitute the *spectrum* of the *bounded* operator $\hat{A}$ and each one of them corresponds to the possible result of a (sharp) measurement of A. Given a general state $|\psi\rangle$, the probability of obtaining a_m as the measurement outcome of A is determined by the Born rule $|\langle\psi|a_m\rangle|^2$. To that effect, one can benefit from the *expansion theorem* which states that any vector $|\psi\rangle \in \mathcal{H}$ has a unique expansion

$$|\psi\rangle = \sum_{i=1}^{n} c_i |a_i\rangle \,; \quad c_i = \langle a_i|\psi\rangle \in \mathbb{C}, \tag{6}$$

[9]Technically, *pure* states as opposed to the *mixed* states

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i| \,; \quad 0 \leq p_i \leq 1,$$

of the more general 'density matrix' formalism of QM. Also, two vectors $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ which differ by a complex factor, that is, $|\psi\rangle = \alpha|\phi\rangle \,; \alpha \in \mathbb{C}$, correspond to the same (pure) state since global phases are immaterial in QM.

to which it follows from Born's rule that the probability of obtaining a_m is $|c_m|^2$. In the absence of external influences (*i.e.*, in a *closed* system) the dynamical evolution of quantum systems is determined by the self-adjoint Hamiltonian operator $\hat{H} : \mathcal{H} \to \mathcal{H}$, such that the time development of any state $|\psi(t)\rangle$ is given by the time-dependent Schrödinger equation

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \, ; \quad \forall \; |\psi(t)\rangle \in \mathcal{H}. \tag{7}$$

Note that, as with the classical dynamical maps $T : \mathcal{S} \to \mathcal{S}$ and the canonical equations in (2), the dynamical law governing closed quantum mechanical systems is a first-order differential equation; therefore, given proper initial conditions, the state $|\psi(t)\rangle$ at any later time is uniquely determined by solving (7). In this sense, QM is as deterministic as Classical Physics. However, in general, the Principle of Thoroughgoing Determination (PTD) fails in the quantum regime, so that there is no way to guarantee a positive or negative stance regarding the proposition (representing property attribution) P_i in QM, for any P_i. To see this, let us take a closer look at the structure of physical observables in quantum theory.

Starting out with equation (5), we define the *projection* operator $\hat{P}_i$ such that

$$\hat{A} = \sum_{i=1}^{n} a_i |a_i\rangle\langle a_i| \equiv \sum_{i=1}^{n} a_i \hat{P}_i. \tag{8}$$

Projection operators $\hat{P}$ are operators that project onto some subspace of the Hilbert space. They satisfy the properties of being self-adjoint and idempotent, that is, $\hat{P}^\dagger = \hat{P}$ and $\hat{P}^2 = \hat{P}$, respectively. As such, their only eigenvalues are manifestly 0 and 1; therefore, they can be understood as propositions about properties. For instance, the operator $\hat{P_m}$ represents an observable whose numerical value is defined to equal unity if the result a_m is obtained whenever A is measured (and defined to equal zero otherwise). Thus, P_m is a positive proposition stating the value a_m to A and, accordingly, $\neg P_m$ is its negation. In Classical Physics, the Boolean lattice L_C guarantees that either P_m or $\neg P_m$ applies (the PTD is in action). Now, let $\hat{Q_j} = |b_j\rangle\langle b_j|$ be a projection operator of the spectral decomposition of a self-adjoint operator $\hat{B}$ corresponding to a physical observable B. Projection operators $\hat{P}_i$ and $\hat{Q}_j$ are said to be *orthogonal* if their commutator amounts to zero, that is, $[\hat{P}_i, \hat{Q}_j] \equiv \hat{P}_i \hat{Q}_j - \hat{Q}_j \hat{P}_i = 0$ for all $|\psi\rangle \in \mathcal{H}$; equivalently, the subspaces onto which they project are orthogonal. In QM, commuting operators correspond to *compatible* observables, that is, properties that can be measured simultaneously; thus, commuting projectors are associated with propositions that can be simultaneously asserted in some sense. But, in general, quantum mechanical operators do not commute, so that their corresponding physical observables are said to be incompatible. In particular, if $[\hat{P}_i, \hat{Q}_j] \neq 0$ then their corresponding propositions P_i and Q_j cannot be jointly asserted in any meaningful way. Equipped with logical operators, the quantum mechanical analog of the map Φ in equation (3) then becomes a mapping from the Borel subset $\mathcal{B}(\mathbb{R})$ onto a complete orthomodular lattice L_Q of *quantum* *logic*, a structure of propositions in which the distributive law from classical propositional calculus no longer holds. As it stands, it becomes difficult to define a quantum mechanical object the same way it was done in Classical Physics. Since a thoroughgoing determination is no longer possible, physical systems can only — if ever — be constituted *incompletely* by means of the restricted set of their objective properties (Mittelstaedt 2009). Roughly speaking, at any given time, *only one half of the classical phase space properties can be meaningfully assigned to a quantum system.*[10] Finally, it will be useful to observe that the corresponding projectors $\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n$ of a self-adjoint operator $\hat{A}$ are pairwise orthogonal so that equation (6) can be rewritten as

$$|\psi\rangle = \sum_{i=1}^{n} |a_i\rangle\langle a_i|\psi\rangle = \hat{\mathbb{1}}|\psi\rangle, \tag{9}$$

that is, the sum of all projection operators $|a_i\rangle\langle a_i|$ constitutes a resolution of the identity in $\mathcal{H}$.

3.2 KS theorem and contextuality

Because the quantum mechanical formalism vastly disagrees with the picture of Classical Physics on what concerns property attribution, one could ask whether there is a way to go beyond the statistical predictions of measurement outcomes[11] and, in a sense, *complete* quantum theory. Historically, 'completion approaches' have been called *hidden-variable* theories because they posit a set of observably occult parameters that would assure definite (preexistent) properties to quantum systems. The question, then, is whether or not physical observables can be interpreted in terms of definite, albeit unknown, actual values pertaining to a quantum system. This was answered negatively by Kochen and Specker in 1967. Let us pave the way to their *no-go theorem* by further developing the posed question.

Given any quantum state of a system with observables A, B and C, is there a way to respectively assign numerical values $v(A)$, $v(B)$ and $v(C)$ to those observables?[12] Classically, there are no conceptual problems in constructing such *valuation functions* since, as we saw in section 2, to each physical observable A there corresponds a function $f_A : \mathcal{S} \to \mathbb{R}$ such that the value of A in the state $s \in \mathcal{S}$ is just the value of f_A at s, that is, $f_A(s) \equiv v_s(A)$, where we defined v_s as being the classical valuation function. Further, let $h : \mathbb{R} \to \mathbb{R}$ be a real-valued function and define a new physical observable $h(A)$ by requiring that its

[10] What we have in mind with such an assertion is the canonical commutation relation $[\hat{q_j}, \hat{p_k}] = i\hbar\,\delta_{jk}$.

[11] If the state of a system is an eigenstate of the operator to which its observable is being measured, then clearly the associated eigenvalue is the uniquely predicted result. But, in general, quantum mechanical probabilities less than unity prevail.

[12] The tacit assumption here is that of *non-contextuality*: if A is compatible with both B and C, but B and C are incompatible observables, the value assigned to A will not depend on whether A is being jointly measured with B or C (Mermim 1993).

corresponding function $f_{h(A)}(s) \equiv h(f_A(s))$ for all s.[13] Employing the definition of v_s, it follows that both physical observables $h(A)$ and A satisfy the *functional composition principle* (*FUNC*)

$$v_s(h(A)) = h(v_s(A))\,; \quad \forall\, s \in \mathcal{S}. \tag{10}$$

Equation (10) is reasonably telling us that the value of a function of a physical observable is equal to the function evaluated on the value of that observable. Going back to QM, apart from the special case where a system is in an eigenstate of an operator, there is not an obvious way to construct a quantum valuation function since, within the minimal interpretation adopted here, it is not assumed that an observable has a value before it is measured.[14] However, there are some reasonable conditions that should be applied in constructing a *global* valuation v on the set of all bounded, self-adjoint operators on $\mathcal{H}$, namely:

(i) the *value-rule* holds, that is, the valuation $v(\hat{A}) \in \mathbb{R}$ belongs to the spectrum of the operator $\hat{A}$;

(ii) the *FUNC* principle holds, that is, $v(h(\hat{A})) = h(v(\hat{A}))$ for any real-valued function h.

These requirements suggest the following. First, if two operators $\hat{A}$ and $\hat{B}$ commute, then the global valuation is additive in the sense that, for all $|\psi\rangle \in \mathcal{H}$,

$$v(\hat{A} + \hat{B}) = v(\hat{A}) + v(\hat{B}). \tag{11}$$

Second, if again $[\hat{A}, \hat{B}] = 0$ for all $|\psi\rangle \in \mathcal{H}$ then

$$v(\hat{A}\hat{B}) = v(\hat{A})v(\hat{B}). \tag{12}$$

From this last equation, let $\hat{A} = \hat{\mathbb{1}}$, then $v(\hat{B}) = v(\hat{\mathbb{1}})v(\hat{B})$, thus, $v(\hat{\mathbb{1}}) = 1$ (provided that $v(\hat{B}) \neq 0$). This becomes interesting if the valuation function is applied to projection operators. Recall that projection operators play the role of propositions in QM and, as such, their eigenvalues are either 0 or 1. Hence, the global valuation of any projection operator $\hat{P}$ is either $v(\hat{P}) = 0$ or $v(\hat{P}) = 1$. Moreover, we saw that a set $\{\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n\}$ containing the projectors of the spectral decomposition of a self-adjoint operator $\hat{A}$ forms a resolution of the identity,

$$\sum_{i=1}^{n} \hat{P}_i = \hat{\mathbb{1}}. \tag{13}$$

It implies at once that

$$v(\hat{\mathbb{1}}) = v\left(\sum_{i=1}^{n} \hat{P}_i\right) = \sum_{i=1}^{n} v(\hat{P}_i) = 1, \tag{14}$$

[13] We have $f_{h(A)} \equiv (h \circ f_A) : \mathcal{S} \to \mathbb{R}$. Thus, $h(A)$ is *defined* by saying that its value in any state s is the result of applying the function h to the value of A (Isham and Butterfield 1998).

[14] Because *quantization schemes* are still an open problem, it is not even obvious if the quantum valuation function should be over the physical quantity A or the operator $\hat{A}$ (see Isham 1995). Here, we take the latter approach.

which means that, whenever we have a collection of n pairwise orthogonal projectors $\hat{P}_i = |a_i\rangle\langle a_i|$, there is a single $\hat{P}_i$ for which $v(\hat{P}_i) = 1$, while $v(\hat{P}_j) = 0$ for all $i \neq j$. There are n different ways of associating the value 1 with only one of these projectors (that is, with one of the vectors $|a_i\rangle$). If we consider other distinct orthogonal bases in $\mathcal{H}$ and assume that the value (1 or 0) associated with that vector is the same, irrespective of the choice of the other basis vectors, then we are led to a contradiction and hence our primary hypothesis, the global valuation function, must be contested. This is, in fact, the KS theorem:

KS theorem. *There is no global valuation function if the Hilbert space $\mathcal{H}$ is such that* $\dim(\mathcal{H}) > 2$.

The KS theorem precludes the existence of global valuation functions whenever the dimension of the Hilbert space $\dim(\mathcal{H})$ is greater than 2. To answer our original question, an interpretation of physical observables in terms of definite, actual values pertaining to a quantum system is problematic since no global valuation exists in order to *globally* assign definite, actual values to projection operators. Instead, a value ascribed to an observable A must depend on some specific *context* from which A is to be considered. In algebraic terms, the valuation function $v\,[\mathcal{A}]$ over the *algebra* $\mathcal{A}$ is (once again) a global valuation if $\mathcal{A}$ is the set of all bounded, self-adjoint operators on $\mathcal{H}$. We say that $\mathcal{C}$ is a context if $\mathcal{C} \subset \mathcal{A}$ is a *commutative subalgebra* generated by the set $\{\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_n\}$ of bounded, self-adjoint operators on $\mathcal{H}$. At last, we define a *local* valuation $v\,[\mathcal{C}]$ as a valuation function over a context $\mathcal{C}$. Yet again, the KS theorem precludes the existence of global valuation functions such as $v\,[\mathcal{A}]$, so that Hilbert spaces with dimension greater than 2 admits local valuations only. This quantum mechanical feature is known as *contextuality*, that is, one can have a set of projectors that commute in a given context, *i.e.*, propositions (or properties) that can be simultaneously asserted (or measured), but that no truth-value can be globally assigned to them in the totality of contexts (see de Barros, Holik, and Krause 2017); hence, quantum theory is said to be contextual. However, whereas an *epistemic* reading of contextuality says that measurement outcomes of an observable A depend on whether another set of physical observables are being jointly measured with A (Peres 2002), an *ontic* reading focuses on the orthodox structure of Hilbert spaces to claim that, regardless of measurements, contexts are bases (or *complete sets of commuting operators*) to which their projectors cannot be interpreted as preexistent properties possessing definite values (de Ronde 2019).

We encourage the reader to consult both the original (Kochen and Specker 1967) and more recent proofs of the theorem. Numerous discussions surrounding contextuality and its implications are amalgamated in the reference (*e.g.*, Isham 1995; Mermim 1993; Amaral and Cunha 2018). In particular, we will now follow the work done by de Ronde and Massri (2016) on invariance of quantum states as it relates to the KS theorem.

3.3 Invariance of quantum states: a corollary to KS

To begin with, let us note that the algebra of observables in Classical Physics is commutative, hence the classical local valuation function always coincides with a global valuation (reflecting the Boolean structure of the lattice L_C and the Principle of Thoroughgoing Determination embedded in it). Moreover, the laws of both classical mechanics and relativity theory are invariant under their respective group of transformations; in particular, nomological and dynamical properties are consistently translated between different frames of reference. As we saw, the fulfillment of these conditions is what underpins classical objectivity and allows one to speak of the state $s \in \mathcal{S}$ as being an actual state of affairs — *ASA*, as it was called. Algebraically, this corresponds to the mathematical feature of consistently pasting together multiple contexts of local valuations into a single global valuation function, to which we call such feature as *value invariance* (*VI*). One can be motivated to specify a *VI* with respect to nomological and dynamical properties, *VINP* and *VIDP*, respectively. If both *VINP* and *VIDP* are satisfied (and hence we get *VI*) then we have an *ASA*. In reality, the invariance of the valuation of both the sets of nomological and dynamical properties of a state $s \in \mathcal{S}$ is, effectively, what reifies an actual state of affairs in the classical picture. In quantum theory, however, the algebra of observables is non-commutative, so that the KS theorem prohibits global valuations from existing and imposes restrictions over a *VI*: even though the *state-vector* $|\psi\rangle \in \mathcal{H}$ is *defined* to be invariant under rotations, it is not a mathematical invariant of the kind needed to provide an interpretation in terms of objects possessing definite physical properties. More specifically, although the invariance of nomological properties (*VINP*) is respected in the formalism, valuations of dynamical magnitudes are not preserved under rotations (failure of *VIDP*) and thus we do not have, in general, *VI* in QM. Therefore, within the quantum mechanical formalism, one cannot reify the vector $|\psi\rangle$ as an *ASA*. This was shown by de Ronde and Massri (2016) through a corollary to the KS theorem which we reproduce next.

Corollary. *If* $\dim(\mathcal{H}) > 2$*, then the* VIDP *of a vector* $|\psi\rangle \in \mathcal{H}$ *is precluded.*

Proof. We refer the reader to the proof in de Ronde and Massri (2016), but the basic idea consists of submitting a vector in Hilbert space to a sequence of rotations. The authors showed that this, in turn, exhausts the local valuation function defined over a *maximal* context.[15] The failure of *VI* leads to the stated result. □

In their work, de Ronde and Massri pointed out that, since invariance of nomological properties is a *necessary* but not a *sufficient* condition to characterize a physical system, one cannot interpret a vector as being an actual individual entity. This is our cue to resume the discussion regarding bundle and

[15] We say that a context $\mathcal{A}$ is maximal if, given a self-adjoint operator $\hat{B}$ such that $\hat{B}\hat{A} = \hat{A}\hat{B}$ for all $\hat{A} \in \mathcal{A}$, then $\hat{B} \in \mathcal{A}$.

substratum theories of objects — we are finally ready to make our case against them.

4 Bundle and substratum theories (reconsidered)

When we left the discussion around bundle and substratum theories in order to explore the machinery of quantum mechanics, we saw that, despite all the problems surrounding the metaphysical dispute, both theories were equally successful in accounting for property possession and for individuality of concrete particulars, agreeing with our pre-theoretic intuition that such entities are individuals and, as such, with the Boolean structure of Classical Physics. This, in turn, created what was called an underdetermination problem of metaphysics by physics, inasmuch as our physical theories were completely silent with respect to their preference of bundle over substratum frameworks, and vice versa. As it happens, the situation in QM is even more dramatic. While it is mostly considered that Classical Physics is at least committed to the notion of individuality of physical systems, objects in quantum theory can either be interpreted in terms of individuals or, unprecedentedly, as *non-individuals* (see again French and Krause 2006, chap. 4).

To see that, recall the PII: it states that there are no numerically distinct indistinguishable objects. Classically, only a weak version of the PII, such as PII(1) (in which intrinsic properties and relations are assignable to an object), can survive the subtleties of classical physical situations. In quantum theory, however, a textbook-example such as two electrons in a box serves to decimate the PII in just a back-of-the-envelope calculation: compose the Hilbert space of two electrons in accordance to equation (4) and then apply the expectation value to *any* observable; the result obtained is invariant under particle permutation. There are no physical means whatsoever to tell which electron is which. Therefore, it is said that both electrons lost their individuality in some sense. Accordingly, physical systems are no longer subject to the Maxwell-Boltzmann statistics; the latter becomes obsolete in QM, and one needs to comply with the formalism by adopting the so-called Fermi-Dirac and Bose-Einstein statistics. To the founding fathers of QM, such as Schrödinger, this was enough to abandon the individuality of quantum mechanical entities and adopt the framework of non-individuals (French and Krause 2006, chap. 4; see also Arenhart 2017). But then, again, that would undermine the metaphysical details discussed in section 2 in the following sense: the failure of PII implies the failure of bundle theories in accounting for the individuality of concrete particulars, but nothing related can be said about the substratum approach to individuality (or perhaps other weaker versions of bundle theories such as the one presented by Rodriguez-Pereyra 2004). Actually, one could even argue that QM exposes the limitations within the strategy of exhausting an object solely into its constitutive properties, vindicating the need for a substratum in order to constitute and individualize concrete particulars (a related claim may be found in French and Krause 2006, chap. 4). Thus, quantum theory takes the underdetermination of metaphysics by physics to an upper level. The space of metaphysical discourse is expanded in a way to cover both metaphysics of individual systems (through substratum theory) and non-individual systems. If Classical Physics made us turn all the way 'back to square one', QM surpassed the rules and changed the game itself. We believe, however, that the new rules are in our favor.

It was stated that, amid the well-known disparities between bundle and substratum theories, both frameworks were inherently committed with the basic operation of property assignment via PTD, that is, given a property P and a particular a, either a possesses P or else it possesses $\neg P$. In the physicist jargon, we were then able to show that such attribute assignment is equivalent to physical theories to which the notion of physical observables (properties) are 'isomorphic' to propositions about physical systems. Hence, we can say that the instantiation of properties by concrete particulars is on equal footing with the assignment of truth-valued propositions describing physical systems. In particular, it was shown that Classical Physics is a fertile soil for both bundle and substratum theories, since the classical formalism contains in it a Boolean structure of definite-valued propositions suited enough to accommodate the PTD. However, as we have seen all throughout this work, such feature is simply another description of what is completely untenable in quantum theory: as a result of the algebra of non-commutative observables, the KS theorem follows from the quantum mechanical formalism itself to preclude a global assignment of preexistent values to projection operators (properties) of physical systems. One cannot assert values to properties of a quantum system because there are no definite values to be asserted in the first place. Our approach aims at the conceptual foundations shared by both bundle and substratum theories, to which one finds the PTD as the common ground they stand. Albeit a solid bedrock in Classical Physics, the PTD, and hence bundle and substratum theories, does not survive the restrictions imposed by QM and, in particular, the KS theorem.

In other words, instead of concentrating on whether distinct versions of the bundle theory could be cooked up to account for indiscernible quantum entities, and whether substratum theories are up to the job, we have advanced an argument to the effect that a fairly neglected principle of both theories (the PTD) fails in quantum theory. That brings both metaphysical theories down in a single stroke, showing clearly that quantum mechanics may be profitably used as a test field for metaphysical theories (a claim advanced, for instance, in Arenhart 2012, Arroyo 2020). Obviously, one may attempt to weaken the PTD in order to account for the indeterminacy of quantum theory, but then, such an account still has to be worked out in details, and it is not clear that what will result will still have the attractions that the original theories had to begin with.

Instead of speculating on how such theories could be modified to resist the KS argument, it is more interesting in this moment to check how deep the result goes. It affects even recent modifications of bundle theory, without having to delve into the details of which version of PII is valid or not. For instance, Rodriguez-Pereyra (2004) made a compelling argument in favor of bundle theory without the need of any PII version; indeed, the author was able to refute the PII within his version of bundle theory. Nevertheless, that's immaterial to how the approach deals with property possession, and the KS kind of argument presented here can be applied to generate trouble to Rodriguez-Pereyra's bundle theory, since it makes use of the very account of attribute assignment that does not survive the results of quantum theory. Equivalently, Jago's *essential bundle theory* (Jago 2018), which attempts to circumvent problems of distinguishing between essential and accidental property attribution in quantum theory will also have problems with the KS argument. In order to account for the distinct modal status of some properties, one still needs to provide for a consistent thorough distribution of truth values for the propositions attributing properties to entities (the PTD), while making a distinction (that Jago is willing to ground) between essential ones and non-essential ones. The fact that some properties are essential while others are not essential does nothing to prevent the argument above from running. Furthermore, Jago's approach requires that spatio-temporal location is a definite property constituting every bundle. Given that spatio-temporal location may be undefined in some contexts in QM, this approach too faces the consequences of the argument developed here.

Notice that this is a general argument against versions of bundle theory and substratum theory that are willing to account for property attribution and composition. The simplicity and explanatory power of such theories make them appealing at first; however, the KS theorem makes for such theories unable to account for quantum entities. This test of such theories needs not discuss the issue of individuality. This, it seems to us, is an advantage, given that the rod for these theories is blocked independently of the problem of individuality, already much discussed in the literature.

5 Conclusion

In this work, we attempted to enrich and potentially enlighten an old discussion regarding the nature of concrete particulars by welcoming into the conversation quantum theory and recent developments on invariance of quantum states as it relates to the Kochen-Specker theorem (as developed by de Ronde and Massri 2016). We have argued that current versions of both bundle theory and substratum theory all adopt a version of what we have called (following Mittelstaedt) the Principle of Thoroughgoing Determination, which requires that a particular either instantiates or does not instantiate any given property P. This principle is vindicated by bundle and substratum theories by the very approach they offer on the constitution of a concrete particular. We have shown that this principle is untenable when QM enters the stage, so that these approaches to particulars are not consistent with quantum theory. Of course, this opens the door for structuralist accounts of quantum entities, but that was not the topic of our discussion.

A further consequence of the result being advanced here concerns metaphysical underdetermination. It was argued, for instance, by French and Krause (2006, chap. 4), that QM underdetermines its metaphysics of particular objects. That is, as we have discussed, quantum theory, when understood as dealing with objects (*i.e.*, as providing for an object-oriented ontology), may be interpreted as dealing with individual objects (whose principle of individuality is provided by substratum theory, and not by bundle theory) or else with non-individual objects (not individuated, of course).[16] By taking into account only the issue of individuality, French and Krause were able to argue that a substratum theory may live with indiscernible quantum entities, given that individuality is provided for by the substratum. However, this focus on the problem of individuality somehow blinded them to the fact that substratum theories (and bundle theories without the PII too, if those were taken into account) are clearly *incompatible* with quantum theory due to the KS theorem. As a result, substratum theories are also not really an option to account for those entities' individuality; that is, those theories are not an option to account for the very objecthood of quantum entities if those items are understood as concrete particulars. If that is correct, then, only the non-individuals interpretation is left as a legitimate metaphysics of concrete particulars. But is it? Well, it all depends on how property possession for non-individuals is accounted for, and non-individuals have a nebulous metaphysics, to say the least. But that is an issue for another discussion.

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